Math 249 Lecture 24 Notes

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1 Composition of Species

1.1 Composition of generating functions

Last lecture, we defined the notion of addition and multiplication of species and saw that

$$\mathcal{F}_{A+B} = \mathcal{F}_A + \mathcal{F}_B,$$

$$\mathcal{F}_{AB} = \mathcal{F}_A \mathcal{F}_B.$$

We want to define composition of species in a way that is compatible with the generating functions. In general, let $F = \sum f_n x^n$ and $G = \sum g_n x^n$ be formal power series. Then $F \circ G = \sum f_n G(x)^n$ is well-defined if $G(0) = 0$.

Definition 1.1. Let F, G be species where $G(\varnothing) = \varnothing$. Then we define the *composition* $F \circ G$ as follows. Partition S into blocks, and take the set of structures with every block having G-structures and the set of blocks having F structures. In other words,

$$(F \circ G)(S) = \coprod_{\{B_1, \dots, B_k\} \in \Pi(S)} F(\{B_1, \dots, B_k\}) \times \prod_i G(B_i).$$

Proposition 1.1. *Let F, G be species where $G(\varnothing) = \varnothing$.*

$$\mathcal{F}_{F \circ G} = \mathcal{F}_F \circ \mathcal{F}_G$$

Proof. Let $(F \circ G)_k$ be the subspecies where $(F \circ G)_k(S)$ is the set of $(F \circ G)_k(S)$ structures with k blocks. Then

$$|(F \circ G)_k(S)| = F([k]) \cdot \frac{1}{k!} |G^k(S)|$$

because $G^k(S)$ is the set of G-structures on S partitioned into k (ordered) blocks, and there are $k!$ orderings of the blocks. We can then compute

$$\mathcal{F}_{F \circ G}(x) = F(\varnothing) + \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} \frac{1}{k!} F([k]) |G^k([n])| \right) \frac{x^n}{n!}$$

$$= F(\varnothing) + \sum_{k=1}^{\infty}\sum_{n=k}^{\infty}\frac{1}{k!}F([k])|G^k([n])|\frac{x^n}{n!}$$

Note that $\mathcal{F}_{G^k}(x)$ only has terms starting at x^k onwards because $|G(\varnothing)| = 0$, so we may identify the inner sum as $\mathcal{F}_{G^k}(x)$.

$$\begin{aligned}
&= F(\varnothing) + \sum_{k=1}^{\infty} F([k])\frac{\mathcal{F}_{G^k}(x)}{k!} \\
&= F(\varnothing) + \sum_{k=1}^{\infty} F([k])\frac{(\mathcal{F}_G(x))^k}{k!} \\
&= (\mathcal{F}_F \circ \mathcal{F}_G)(x).
\end{aligned}$$

□

1.2 Examples of species composition

1.2.1 Partitions and Stirling numbers

Example 1.1. Let $\Pi(S) = \{\text{partitions of } S\}$. Then

$$\Pi \cong E \circ (E - X_0),$$

which gives us that

$$\Pi(x) = \mathcal{F}_E \circ \mathcal{F}_{E-X_0} = e^{e^x - 1}.$$

Example 1.2. Look at the Stirling number $S(n,k)$, the number of partitions of $[n]$ into k blocks. We saw earlier that $k!S(n,k) = \left|(E - X_0)^k([n])\right|$. We had

$$\sum_{n=0}^{\infty} S(n,k)\frac{x^n}{n!} = \frac{(e^x - 1)^k}{k!}.$$

Summing over k would give us the same result as in the previous example, that

$$\Pi(x) = e^{e^x - 1}.$$

Consider E, the trivial species, with weight t^k for the unique structure on S if $|S| = k$. Then

$$E(x;t) = \sum_n t^n \frac{x^n}{n!} = e^{tx}.$$

We can use this to say

$$E(x;t) \circ_x (E(x) - 1) = e^{tx} \circ_x (e^x - 1) = e^{t(e^x - 1)},$$

and so the weighted generating function for Stirling numbers is

$$\sum_{n,k} S(n,k)t^k \frac{x^n}{n!} = e^{t(e^x - 1)}.$$

1.2.2 Permutations and cyclic orderings

Example 1.3. Let $P(S) = \{\text{permutations of } S\}$ and $C(S) = \{\text{cyclic orderings of } S\}$. Then $P([n]) = n!$, and $C([n]) = (n-1)!$, where by convention, there is no cyclic ordering on the empty set. The statement that permutations have disjoint cycle structures gives us

$$P \cong E \circ C.$$

So we get that

$$P(x) = e^{C(x)}.$$

We already know that $P(x) = \frac{1}{1-x}$, so we get

$$C(x) = -\log(1-x).$$

Alternatively, we could use the fact that $C([n]) = (n-1)!$, which gives us that

$$C(x) = \sum_{n=1}^{\infty}(n-1)!\frac{x^n}{n!} = \sum_{n=1}^{\infty}\frac{x^n}{n} = -\log(1-x).$$

We can use this information to get other interesting generating functions. Consider the species C_{even} which only counts the cyclic orderings of sets of even numbered elements. Then

$$C_{\text{even}}(x) = \sum_{n=1}^{\infty}\frac{(x^2)^n}{2n} = -\frac{1}{2}\log(1-x^2).$$

Then the number of permutations into cycles of even lengths is $P_{\text{even}} \cong E \circ C_{\text{even}}$, and we get

$$P_{\text{even}}(x) = e^{C_{\text{even}}(x)} = (1-x^2)^{-1/2}.$$

Example 1.4. Weight a permutation $\sigma \in S_n$ with $P_{\gamma(\sigma)} = p_1^{|\{\text{1-cycles}\}|}p_2^{|\{\text{2-cycles}\}|}\cdots$. For now, treat the p_k as new variables, but they will end up being symmetric functions. The exponential generating function with all these weights is

$$P(x;p_1,p_2,\dots) = \sum_n\left(\sum_{\sigma\in S_n}p_{\gamma(\sigma)}\right)\frac{x^n}{n!}.$$

For example, for $n=3$, the inner term is $p_1^3 + 3p_2p_1 + 2p_3$. We say that this is equal to $e^{C(x;p_1,p_2\dots)}$, where

$$C(x;p_1,p_2\dots) = \sum_{n=1}^{\infty}p_n\frac{x^n}{n}$$

is the generating function for cyclic orderings weighting the cycle structure. We have

$$P(x;p_1,p_2,\dots) = \exp\left(\sum_{n=1}^{\infty}p_n\frac{x^n}{n}\right) = \prod_{n=1}^{\infty}\exp\left(p_n\frac{x^n}{n}\right)$$

$$
\begin{aligned}
&= \prod_{n=1}^{\infty} \sum_{r_n=0}^{\infty} p_n^{r_n} \frac{x^{r_n n}}{n^{r_n} r_n!} \\
&= \sum_{r_1, r_2, \dots} (p_1^{r_1} p_2^{r_2} \cdots) \frac{x^{r_1 + 2r_2 + \cdots}}{\prod_k r_k! n^{r_k}} \\
&= \sum_{n=0}^{\infty} \sum_{\substack{|\lambda|=n \\ \lambda=(1^{r_1}, 2^{r_2}, \dots)}} \frac{p_\lambda n!}{z_\lambda} \frac{x^n}{n!}.
\end{aligned}
$$

This gives us that

$$\sum_{|\lambda|=n} p_{\gamma(\sigma)} = \frac{n!}{z_\lambda} p_\lambda$$

is the number of permutations $\sigma \in S_n$ such that $\gamma(\sigma) = \lambda$.

If we view p_k as the k-th power sum symmetric function, then

$$\frac{1}{n!} \sum_{\sigma \in S_n} p_{\gamma(\sigma)} = F(\mathbb{1}) = h_n,$$

so

$$P(x; p_1, p_2, \dots) = H(x) = \sum_{n=1}^{\infty} h_n x^n,$$

the generating function for the homogeneous symmetric functions.